SOME CLASSAS OF FULL STABLITY BANACH ALGEBRA MODULES

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Abstract: The research is an article that teaches some classes of fully stable Banach - Å modules. By using Unital algebra studies the properties and characterizations of all classes of fully stable Banach - Å modules. All the results are existing, and they've been listed to complete the requested information.

Keywords: Fully Stable Banach Algebra Modules, Fully Stable Banach Algebra Modules Relative to an ideal and Srongly fully stable Banach Algebra Modules Relative to an ideal

1. Introduction

Banach algebra theory (BÅ) is a mathematical abstraction. When abstract ideas and structures were first introduced in the early 20th century, both the language and the practice of mathematics underwent radical change. Let Å be a non-empty set, the set A is said to be algebra if (1) $(Å, +, .)$ is vector space over a field F, (2) $(Å, +, .)$ is a ring, and (3) $(a)b = (ab) = a(b), \forall\, a, b \in Å$ [1]. In [2] S. Burris and

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H. P. Sankappanavar show that a ring R is an algebra $< R, +, \cdot, -, 0 >$ where $+$ and $\cdot$ are binary, $-$ is unary, and 0 is nullary satisfying that $< R, +, -, 0 >$ is an abelian group, $< R, . >$ is a semigroup and $x.y + z = x.y + (x.z)$, $x + y.z = x.z + (y + z)$. The following is a definition of the Banach left Å- module that may be found in [3]: let Å be algebra, a Banach space E is called a Banach left Å-module if E is a left module over algebra Å and $\|a.x\| \|a\| \|x\| a \in$ Å, $x \in E$ [3]. In [4], a multiplier (homomorphism) means a map from a left Banach Å-module X into a left Banach A-module Y (A is not necessarily commutative) if it satisfies $Ta.x = a.Tx$, ∀ a ∈ Å, x ∈X. A submodule N of an R-module M is called stable, if $f(N) \subseteq N$ for each R-homomorphism from N to M. In case each submodule of M is stable, then M will be called a fully stable module [5]. A fully stable Banach Å-module M is called fully stable Banach Å-module if for every submodule N of M and for each multiplier $\theta : N \longrightarrow M$ satisfies $N \subseteq N$ [6]. A Banach Å-module M is deemed a fully stable Banach A-module relative to an ideal K of Å if for every submodule N of M and for each multiplier $\theta : N \longrightarrow M$ satisfies $N \subseteq N + KM$. For a nonempty subset M in a left Banach A-module X, the annihilator annÅ(M) of M is $ann_{Å}M = a$Å; $a.x = 0, \forall x \in M$ [7]. In [6], Let X be a Banach Å-module, $Nx = \{nx |\ n \in N,\ x \in X\}$ and $P_y = \{py |\ p \in P, y \in X\}$, $ann_A Nx = \{a \in A,\ a.nx = 0,\ \forall\ nx \in Nx\}$ and $ann_A Py = \{a \in A,\ a.py = 0,\ \forall py \in Py\}$. Recall that for $n \in N$, a left Banach Å-module X is said to be n-generated if there exists $x_1, \dots, x_n \in X$ s.t each $x \in X$ can be represented as $x = k = 1a_k.x_k$ for some $a_1, \dots, a_n \in$ Å . A cyclic module is just a 1-generated module [8The R-module If each of M's submodules has the form IM for a certain ideal I of R, then M is a multiplication module [9]. It is examined how fully stable Banach modules compare to ideals in terms of their constituents.

2. Fully Stabel Banach Å-Modules

Definition 2.1: “Put X is a B-Å-M, if ∀ submodule N of X and for each multiplier $f : N \longrightarrow X \ni f(N) \subseteq N$, then X is said to be F-S-B- Å-M” [10].

The proof of proposition (2.3) is in [10].

Definition 2.2: "A B-Å-M X is said to be satisfy B-C (Baer criterion) if each submodule of X satisfies B-C, this mean that ∀ submodule N of X and A -multiplier f : N ⟶ X, ∃ an element a ∈ A such that f(n) = an ∀ n ∈ N".

Proposition 2.3:

1- X is F-S-B- Å-M iff ∀ x, y ∈ X and N_x, K_y subsets of X , y ∉ N_x implies that $\mathrm{ann}_{Å}\, N_x \not\subseteq \mathrm{ann}_{Å}\, K_y$.

2- Let X be a F-S-B Å-M. If ∀ x, y ∈ X, $\mathrm{ann}_{Å} K_y = \mathrm{ann}_{Å} N_x$ then $N_x = K_y$.

Proposition2.4: "Put X is a B-Å-M. Then $\mathrm{ann}_X(\mathrm{ann}_{Å}(N_x)) = N_x$, ∀ x ∈ X"B-C, iff holds for cyclic submodules of X".

Proof: "Assuming that B-C holds, and let $y \in \mathrm{ann}_X(\mathrm{ann}_{Å}(N_x))$. Define $f : N_x \to X$ by $f(a.nx) = a.ky$, ∀ a ∈ Å . Put $a_1.nx = a_2.nx$, hence $(a_1 - a_2)nx = 0$, $a_1 - a_2 \in \mathrm{ann}_{Å}(N_x)$, $(a_1 - a_2) \in \mathrm{ann}_{Å}Ky$, then $(a_1 - a_2)ky = 0$, thus $a_1 ky = a_2 ky$, hence f is well defined. Clearly f is Å -multiplier, and by assumption, ∃ t ∈ Å ∋ $tmx = f(mx)$, $\forall mx \in N_x$ .Then in particular, $ky = f(nx) = tnx \in N_x$ $\mathrm{ann}_X(\mathrm{ann}_{Å}(N_x)) \supseteq N_x$ hence $\mathrm{ann}_X(\mathrm{ann}_{Å}(N_x)) = N_x$. On another direction, assuming that $\mathrm{ann}_X(\mathrm{ann}_{Å}(N_x)) = N_x$ $\forall N_x \in X$, and for each A -multiplier $f : N_x \to X$, $s \in \mathrm{ann}_A(N_x)$, then we have $sf(nx) = f(snx = 0)$. Thus $nx \in \mathrm{ann}_X(\mathrm{ann}_A(N_x)) = N_x$ then $nx = t\,nx$, for some t ∈ A, thus Baer criterion is holding".

Corollary 2.5: "X is F-S-B Å- M iff $\mathrm{ann}_X(\mathrm{ann}_A(N_x)) = N_x$ $\forall x \in X$".

3. Fully Stabel Banach ALgebra Modules Relative To Ideal

Definition 3.1: "Let X be a B Å- M, X is called F-S-B Å- M-R-I K of Å, if ∀ submodule N of X and for each multiplier θ : N ⟶ X satisfies N ⊆ N + KX. Clearly, every F-S-B- Å-M is a F-S-B Å-M-R-I" [11].

Proposition 3.2: "X is F-S-B Å- M-R-I iff ∀x, y ∈ X and N_x, Ky are subsets of X, y ∉ N_x + KX then $\mathrm{ann}_{Å}N_x \not\subseteq \mathrm{ann}_{Å}Py$ ".

Proof: "assume that X is a F-S-B Å- M-R-I, K of Å, ∃ x, y ∈ X s.t y ∉ N_x + KX and $\mathrm{ann}_{Å}N_x \subseteq \mathrm{ann}_{Å}\, P_y$. Now, define $\alpha : < N_x > \longrightarrow X$ by $\alpha(a.nx) = a.py$,

$\forall$a $\in$ Å, if a.nx = 0 → a $\in$ $\mathrm{ann}_{Å}(N_x) \subseteq \mathrm{ann}_{Å}$Py. This yields a.py = 0, hence α is well defined, clear α is a multiplier, because X is F-S-B Å- M-R-I, ∃ an element t $\in$ A ∋ mx = tmx + w, $\forall$mx $\in N_x$, w $\in$ KX. Particularly, py = α(nx) = tnx + w $\in$ N_x +KX, which is a contradiction. Hence X is a F-S-B Å- M-R-I. On anther direction, assuming that ∃ a subset N_x of X and a multiplier f : $< N_x > \longrightarrow$ X s.t $N_x \nsubseteq N_x$ + KX then ∃ an element mx $\in N_x$ ∋ mx $\notin N_x$ + KX. Now, let s∈$\mathrm{ann}_{Å}(N_x)$; therefore, snx = 0, sf(mx) = f(smx) = f(stnx) = f(tsnx) = f(0) = 0. Hence $\mathrm{ann}_{Å}(N_x)$= $\mathrm{ann}_{Å}$(f(mx)), and this is a contradiction".

Corollary 3.3: "Let X be a F-S-B Å- M-R-I, K of Å. Then ∀ x, y $\in$ X, and $\mathrm{ann}_{Å}$ (P_y) = $\mathrm{ann}_{A}(N_x)$ implies that N_x + KX = Py+KX".

Definition 3.4: " A B-Å-M, X is said to be satisfying B-C-R-I (Baer criterion relative to an ideal) K of Å, if each submodule of X satisfies B-C-R-I, i.e. ∀ submodule N of X and Å - multiplier f: N ⟶ X, ∃ an element a $\in$ A s.t n − an $\in$ KX, ∀ n $\in$ N".

Another characterization of F-S-B-Å-M-R-I is given in proposition (3.5) and corollary (3.6). For the proof see [11].

Proposition 3.5: "Put X is a B-Å- M. Then the B-C-R-I holds for cyclic submodules of X iff $\mathrm{ann}_{X}\left(\mathrm{ann}_{Å}(N_x)\right) = N_x +$ KX, ∀x $\in$ X".

Corollary 3.6: "X is F-S-B-Å-M-R-I, K $\in$ Å iff $\mathrm{ann}_{X}\mathrm{ann}_{A}(N_x) = N_x +$ KX, ∀ x $\in$ X".

Definition 3.7: "A B-Å-M, X is said to be a multiplication Å-module if each 1-generated submodule of X is of the form KX for some ideal K∈A".

The relation between full stability and full stability relative to an ideal for Banach algebra modules is discussed the following proposition.

Proposition 3.8: "Let X be a multiplication Banach Å -module. X is F-S-B-Å-M iff it is F-S-B-Å-M-R-I, K of Å".

Proof: It is clear [12]

The quasi α-injective concept is introduced in the following definition.

Definition 3.9: "Let Å be a unital Banach algebra, $\alpha > 1$. An Å-M, X is said to be quasi α-injective if, φ : N ⟶ X is an Å-M homomorphism s.t $\|\varphi\| \leq 1$, ∃ Å-M, homomorphism $\theta: X \longrightarrow X$, s.t $\theta \circ i = \varphi$, $\|\theta\| \leq \alpha$, where i is an isometry from submodule N of X. Then X is said to be quasi injective if it is quasi α - injective for some α".

Definition 3.10: "Let A be a unital Banach algebra and let $\alpha > 1$. An A-module X is called quasi α-injective relative to an ideal K of A if, $\varphi : N \longrightarrow X$ is A-module homomorphism such that $\|\varphi\| \leq 1$, ∃ A-module homomorphism θ: X ⟶X, s.t $(\theta \circ i)(n) - \varphi(n) \in KX$ and $\|\theta\| \leq \alpha$, where i is an isometry from a submodule N of X to X. Then X is quasi injective relative to ideal if it is quasi α – injective relative to ideal for some α".

The relation between the quasi α-injective B-Å-M-R-I and F-S- Å –R-I, K of Å is shown in the following proposition.

Proposition 3.11: "If X is F- S- B- Å-M-R-I, then X is quasi injective B- Å-M-R-I".

Proof: It is clear

4. Strongly Fully Stabel Algebra Modules Relative To Ideal

Definition 4.1: "Put X is a B-Å-M, X is called S-F-S-B-Å-M-R-I, K of Å, if ∀ multiplier f : N ⟶ X and for each submodule N of X, ∋ N ∩ KX ⊇ f(N). Clearly, every F-S-B- Å-M is F-S-B- Å–R-I. Moreover, every F-S-B-Å-M is S-F-S-B-Å-M-R-I, therefore X is S-F-S-B-Å-M-R-I, if and if for each multiplier f :L⟶X for every 1-generated submodule L of X and ∋ f(L)⊆ L∩KX . Put X is a B-Å-M and K is a non-zero ideal in Å. If M is F-S-B-Å-M and KX = X then X is S-F-S-B-Å-M-R-I, K, since Å -homomorphism f : N → X, f(N) ⊆ N = N ∩ X = N ∩ KX for every 1-generated submodule N of X ".

Another characterization of S-F-S-B-Å-M-R-I is given in the following proposition. The proof of the proposition is in [13].

Proposition 4.2: "X is S-F-S-B-Å-M-R-I iff Nx, Py are subsets of X, ∀ x, y ∈ X, y ∉ Nx ∩ KX implicate that $ann_{Å}(Nx) \not\subset ann_{Å}(Py)$".

Corollary 4.3: "Let X be a S-F-S-B-Å-M-R-I, K of A. Then ∀ x, y ∈ X, $ann_{Å}(N_x) = ann_{Å}(P_y)$ implicate that $KX \cap N_x = KX \cap P_y$".

Proof: "Assuming ∃ x, y in X ∋ $ann_{Å}Nx = ann_{Å}(Py)$ and KX ∩ Nx = KX ∩ PyX. Thus ∃ zx in Nx and not in Py. Therefore $ann_{Å}(Py) \not\subseteq ann_{Å}(Zx)$, by proposition (4.2) but $\mathrm{ann}_{Å}(\mathrm{Zx}) \supseteq \mathrm{ann}_{Å}(\mathrm{Nx})$. Hence $\mathrm{ann}_{Å}(\mathrm{Py}) \nsubseteq \mathrm{ann}_{Å}(\mathrm{Nx})$, contradiction".

Definition 4.4: "Pure submodule is a submodule N of B-Å-M satisfies $N \cap KX = KN$ for all ideal K of Å".

Proposition 4.5: "Every pure submodule of a S-F-S-B-Å-M-R-I, K of Å X is S-F-S-B-Å-M-R-I".

Proof: "let N be a pure submodule of X. ∀ submodule L of N and a multiplier $f : L \rightarrow N$, put $g = i \circ f : L \rightarrow X$ (where i is the inclusion mapping from N to X), then by assumption $f(L) = g(L) \subseteq KX$, and since $f(L) \subseteq N$. Hence $f(L) \subseteq L \cap KX \cap N$. Since N is pure submodule of X then $N \cap KX = KN$, for each ideal K of Å, therefore $f(L) \subseteq L \cap KN$. Thus N is S-F-S-B-Å-M-R-I, K".

Definition 4.6: "A B-Å-M, X is said to be satisfying the B-C-R-I (Baer criterion relative to an ideal) K of Å, if ∀ submodule of X satisfies B-C-R-I. i.e., ∀ 1-generated submodule N of X and Å- multiplier $f : N \longrightarrow X$, ∃ an element $a \in A$ s.t $f(n) = an \in KX$, $\forall\, n \in N$".

Another characterization of S-F-S-B-Å-M-R-I is given in the following proposition and its corollary.

Proposition 4.7: "Put X is a B-Å-M. Then B-C-R-I holds for 1-generated submodules of X if and only if $\mathrm{ann}_{X}(\mathrm{ann}_{Å}(\mathrm{Nx})) \subseteq \mathrm{Nx} \cap KX$, $\forall x \in X$".

Proof: It is clear [13]

Corollary 4.8: "X is S-F-S-B-Å-M-R-I, K iff $\mathrm{ann}_{X}\big(\mathrm{ann}_{Å}(\mathrm{Nx})\big) \subseteq \mathrm{Nx} \cap KX$, $\forall\, x \in X$".

Definition 4.9: "Let Å be a unital Banach algebra. An Å- module X is said to be a S-Q-α- IN-R-I (strongly quasi α-injective relative to an ideal K of Å, if $\varphi : N \longrightarrow X$ is Å-M homomorphism (multiplier) s.t $\|\varphi\| \leq 1$, ∃ an Å-M homomorphism (multiplier) $\theta : X \longrightarrow X$, s.t $(\theta \circ i)(n) = \varphi(n) \in KX$ and $\|\theta\| \leq \alpha$, where i is an isometry from a submodule N of X to X. Therefore X is S-Q-IN-B-Å-M-R-I if it is S-Q-α- IN-R-I for some α".

The relation between S-Q-α- IN-R-I B- Å-M-R-I and S-F-S-B-Å-M-R-I, K is given in the following proposition.

Proposition4.10: "Put X is a B-Å-M, K is a non-zero ideal of Å. X is S-Q-IN-B-Å-M-R-I, if X is S-F-S-B-Å-M-R-I".

Proof: It is clear [13].

5. Conclusions

The conclusions that may be pointed throughout this study some classes of fully stable Banach algebra modules and prove some properties are listed as follows:

1. X is F-S-B- Å-M iff ∀ x,y∈X and Nx, Ky subsets of X, y∉ Nx implies that ann_Å N_x⊄ann_Å K_y.
2. Put X is a B-Å-M. Then $\mathrm{ann}_{\mathrm{X}}\left(\mathrm{ann}_{\mathrm{Å}}(N_x)\right) = N_x$, ∀ x ∈ X"B-C, iff holds for cyclic submodules of X.
3. Let X be a multiplication Banach Å -module. X is F-S-B-Å-M iff it is F-S-B-Å-M-R-I, K of Å.
4. Put X is a B-Å-M. Then B-C-R-I holds for 1-generated submodules of X iff $\mathrm{ann}_{\mathrm{X}}(\mathrm{ann}_{\mathrm{Å}}(\mathrm{Nx})) \subseteq \mathrm{Nx} \cap \mathrm{KX}$, ∀x ∈ X.

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بعض اصناف من مقسات بناخ الجبرا تامة الاستقرارية

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المستخلص: يلكن Aالجبرا يمتلك عنصر محايد، في هذا العمل تم عرض المفاهيم التالية مقاسات بناخ الجبرا تامة الاستقرارية و مقاسات بناخ الجبر تامة الاستقرارية بالنسبة الى مثالي ومقاسات بناخ الجبرا تامة الاستقرارية بقوة بالنسبة الى مثالي حيث تم عرض بعض صفات هذه المفاهيم وبعض العلاقات التي توضح ربطها بمفاهيم اخرى.

الكلمات المفتاحية: مقاسات بناخ الجبرا تامة الاستقرارية، مقاسات بناخ الجبرا تامة الاستقرارية بالنسبة الى مثالي و مقاسات بناخ الجبرا تامة الاستقرارية بقوة بالنسبة الى مثالي

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