Review of Bounded Linear Operators of Modules

Shireen O. Dakheel[1]
shireeno_math@csw.uobaghdad.edu.iq
Samira Naji Kadim[2]
samirank_math@csw.uobaghdad.edu.iq
Uhood S. AL-Hussani
mailto:uhood.s@sc.uobaghdad.edu.iq
Muna Jasim Mohammed Ali[3]
samirank_math@csw.uobaghdad.edu.iq

Abstract

The aim of our work is review of modules and bounded linear operators (B.L.O). Assume that V is a vector space (v.s.) over a field F. Put T is a linear operator. Put $\mathrm{R} = \mathrm{F}[\mathrm{x}]$ is the ring of polynomials in x with coefficients in F. Define $\emptyset: R \times V \longrightarrow V$ by $\emptyset(P, v) = P(T)v = P.v$.. That $\emptyset$ makes V a left $R-$ module denoted V_T. The generalization of this concept have been introduced, put V is a normed space over a field F, put T is a B. L. O. , and assume that $\mathrm{R} = \mathrm{F}[\mathrm{x}, \mathrm{y}]$ is the ring of polynomials in x, y with coefficients in F. Define $\Psi: R \times V \rightarrow V$ by $\Psi(P, v) = P.v = P(T, T^*)v$. V_{T,T^*} is module. Some properties of thes concepts have been studied.

Key Words: Bounded Linear Operators, Polynomaial Rings, Normed Space and Modules

1-Introduction

Suppose that V is a v. s. and T is a L.O . Put $\mathrm{R} = \mathrm{F}[\mathrm{x}]$. Define $\emptyset: R \times V \longrightarrow V$ by $\emptyset(P, v) = P.v = P(T)v$. V_T is a left $R-$module. The form of every element in V_T have been introduced, if $S = \{V_j : j \in \wedge\}$ is a basis for V , then each element of V_T .can be written as $\sum_{i=0}^{n} \sum_{j\in\wedge} c_{ij} T^i v_j$,where $c_{ij} \in F$. $\sum_{j\in\wedge}$ is the sum is taken over a finite subset of $\wedge$. An operator S is called that similar to the operator

T if there exists an invertible operator A such that $ASA^{-1} = T$.[1]. Put T and S is two L.O .Then V_S is isomorphic to V_T if and only if S is similar to T. V_I is a f. g R −module if and only if V is a f. d. vector space. This relation shows that two sides are true where the operator is the identity operator. Put V is a f. d. v. s., and T be an operator on V ,then V_T is a f. g. R- module. An operator T is said to be of finite rank if its range $T(V)$is f. d.. It is shown that if V is a f. d. v. s. , then V_T is a f. g. R − module. Also if V is f. d. v. s., and T is any operator on V, then TV is f. d.. Hence T is of finite rank [1]. If T is of finite rank, and V_T is f. g, then V is f. d.. An R −module M is said to be Noetherian if for every ascending chian $A_1 \hookrightarrow A_2 \hookrightarrow A_3 \hookrightarrow \cdots$ of submodules of M is stationary. [2] . If V is a f. d. v. s. , and T is an operator on V , then V_T is Noetherian R −module [3]. An R −module M is said to be Artinian if for every d. c. $A_1 \hookleftarrow A_2 \hookleftarrow A_3 \hookleftarrow \cdots$ of submodules of M is stationary [2]. If V is a f. d. v. s., and T is an operator on , then V_T is an Artinian R −module [3] . Let $T: V \longrightarrow V$ be an operator. $v \epsilon V$ is said to be an algebraic element (or T-algebraic) if there exists a non zero polynomial $P \epsilon R$ such that $0.T = P(T)v$ is said to be algebraic if there exists $P \neq 0$ in R such that $P(T)v = 0$ for all $v \in V$ [4]. Let $T: V \rightarrow V$ be an operator, and $A = A(T)$ be the set of all T −algebraic elements. Then A is a subspace of V [3] .Let R be a ring , and let M be an R −module. An element m ∈ M is said to be a torsion element if there exists $r \neq 0$ in R such that $rm = 0$, M is called a torsion R −module if every element in M is a torsion element. There is a relation between the T −algebraic elements and the torsion elements of V_T. this relation is studied in the next proposition . [4]. Let T be an operator on V , then $A_T = \tau(V_T)$ [5] . An R −module M is called faithful if for all $r \in R[rM = 0 \Rightarrow r = 0]$ [6]. V_T is faithful R −module if and only if T is not an algebraic operator. Recall that a subspace $W$of a vector space

V is an invariant subspace of V under T if $Tw \in W$ for all $w \in W$, where T is an operator on the $F -$ vector space V.[7]. Finally, the module of the Unilateral shift operator is given , and show that H_U is a free R –module. Let $U: l_2(\mathbb{R}) \rightarrow l_2(\mathbb{R})$ be the operator defined by $U(x_1, x_2, \dots) = U(0, x_1, x_2, \dots.)$.[7]. This operator called the Unilateral shift operator. Let $H = \{X = (x_1, x_2, \dots) \in l_2(\mathbb{R}) : \exists K \in \mathbb{N}$ such that $x_j = 0\ \forall j > K\}$. H is a subspace of $l_2(\mathbb{R})$. Moreover, if $X = (x_1, x_2, \dots, x_r, 0, \dots) \in H$, then $UX = (0, x_1, \dots, x_r, 0,0, \dots) \in H$. Thus H is an invariant subspace of $l_2(\mathbb{R})$. Hence we can consider $U: H \longrightarrow H$ and H_U is defined . Let $S = \{e_K = (x_1, x_2, \dots): x_K = 1, x_j = 0$ for all $j \neq K, K \in \mathbb{N}\}$, if $= (y_1, y_n, \dots y_n, 0,0, \dots) \in H$, then $Y = \sum_{i=0}^{n} y_i e_i$ .Thus the set S generates H .It is clear that the elements $e_1, e_2, \dots, e_n, \dots$ are linearly independent .Hence the set S is a basis for H[3]. A left R −module M is called cyclic if M is just a 1- generated one this mean that $[M = Rx]$ for some x in M. [4]. Let U be the Unilateral shift operator on H. Then H_U is a cyclic faithful R −module. Hence a free R −module . [3].

2-Modules and Bounded Linear Operators

The aim of this section is introducing a left $R -$module by extending the polynomials ring in to two variables .We illustrate this module through some examples, and prove some propositions. We limit some definitions and remarks need it in this section. All results have been studied in [6]

Definition 2-1 (Normed space) [8]

A normed space X is a vector space with a norm defined on it. Here a norm on a vector space X is a real− valued function on X whose value at an $x \in X$ is denoted by $\|x\|$, called norm of x. Satisfies the following conditions:

1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ for all $x \in X$

2) $\|\alpha x\| = |\lambda|\|x\|$, α is scalar

3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

$(X, \| \ \|)$ is called a normed space .

Definition 2-2: (Inner product space) [8]

An inner product space is a vector space X with an inner product on X. Here an inner product on X is a mapping $< , >$ of $X \times X$ into the scalar field $K(K = \mathbb{R}$ or $K = \mathbb{C})$ of X. Satisfies the following conditions :

1) $< x, x > \geq 0$ and $< x, x >= 0$ if and only if $x = 0$ for all $x \in X$

2) $< x + y, z >=< x, z > +< y, z >$ for all $x, y, z \in X$.

3)$< \alpha x, y > = \alpha < x, y >$ for $x, y \in X$, $\alpha \in K$

4)$< x, y > =\overline{< y, x >}$ for all $x, y \in X$.

$(X, < , >)$ is called an inner product space. An inner product space which is complete called Hilbert space.

Definition 2-3: (Linear operator) [8]

A mapping T from a normed space X into a normed space Y is called an operator, and T is linear operator if it satisfies the following conditions:

1) $T(x + y) = Tx + Ty$ for all $x, y \in X$

2) $T(\alpha x) = \alpha Tx$ for all $x \in X$, α is scalar .

Definition 2-4 (Bounded linear operator) [8]

Let X and Y be normed spaces, and let T from a normed space X into a normed space Y a linear operator. The operator T is said to be bounded if there is a real number c such that $\|Tx\| \leq c\|x\|$ for all $x \in X$.

Definition 2-5 : (Hilbert− adjoint operator T^*) [8]

Let $T: H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the Hilbert− adjoint operator T^* of T is the operator

$T^*: H_2 \longrightarrow H_1$ such that for all $x \in H_1$ and $y \in H_2$

$< Tx, y > =< x, T^*y >$.

Definition 2-6 : (Self –adjoint operator) [8]

A bounded linear operator $T: H \longrightarrow H$ on a Hilbert space H is said to be self $-$adjoint operator if $T = T^*$.

Definition 2-7 (Normal operator)[8]

A bounded linear operator $T: H \longrightarrow H$ on a Hilbert space H is said to be normal operator if $TT^* = T^*T$.

Definition 2-8:_ (Bilateral shift operator)[3]

Let $B: l_2(\mathbb{R}) \longrightarrow l_2(\mathbb{R})$ be defined by $B(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$.

B is called the Bilateral shift operator .

Remarks 2-9: [3] 1. $B_{e_i} = e_{i-1}$ for all $i > 1$, and $B_{e_1} = 0$

2. for each $n \in \mathbb{N}$, $B^n e_k = \begin{cases} e_{k-n} & if\ k > n \\ 0 & if\ \ k \leq n \end{cases}$

Remark 2-10 :[3] The Bilateral shift operator on $l_2(\mathbb{R})$ is the adjoint of the Unilateral shift operator.

Definition 2-11: An operator T in B(H) is called binormal, if $(T^*T)(TT^*) = (TT^*)(T^*T)$ [9]

Definition 2-12 : An operator T in B(H) is called hyponormal, if $T^*T \leq TT^*$

Definition 2-13 : An operator T on H is said to be M-hyponormal operator if there exists a real number M such that ||(T - zI)*x ||≤ M ||(T – zI)x ||for all x in H and for every complex number z ,[11]

Definition 2-14 : (Left $F[x, y]$ $-$Module)[2]

Let V be a normed space over a field F, let T be a bounded linear operator acting on the elements of V on the left , and let $\mathrm{R} = \mathrm{F}[\mathrm{x}, \mathrm{y}]$ be the ring of polynomials in

x, y with coefficients in F. We define $\Psi: R \times V \to V$ by $\Psi(P, v) = P.v = P(T, T^*)v$ .i.e $P(x, y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j$, $a_{ij} \in F$.

It is clear that Ψ makes V a left R-module.We denote this module by V_{T,T^*} and call it the associated R −module of .

In[2] shows that $F[x, y] = (F[x])[y]$, so In the following proposition we introduce the form of each element of V_{T,T^*}

Proposition 2-15 : If S=$\{v_l : l \in \Lambda\}$ is a basis for V. then each element of V_{T,T^*} can be written in the form $\sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} T^i {T^*}^j v_l$,where $c_{il} \in F$.

The symbol $\sum_{l\in\Lambda}$ means that the sum is taken over a finite subset of Λ.

Proof:- Let $w \in V_{T,T^*}$, then $w = \sum_{k=1}^{m'} P_k \cdot w_k$

Where $P_k(x, y) = \sum_{j=0}^{m} (P_k(x))\, y^j$, $P_k(x) = \sum_{i=0}^{n_k} a_{ik} x^i$, $P_k(x, y) = \sum_{j=0}^{m} (\sum_{i=0}^{n_k} a_{ik} x^i)\, y^j \in R$, $w_K = \sum_{l\in\Lambda} b_{Kl} v_l \in V$.

Then $w = \sum_{k=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n_k} a_{ik} T^i {T^*}^j (\sum_{l\in\Lambda} b_{kl} v_l)$

$= \sum_{k=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n_k} T^i {T^*}^j (\sum_{l\in\Lambda} a_{ik} b_{kl} v_l)$

Let $n = \max\{n_1, n_2, \cdots, n_m\}$, and let $a_{ik} = 0, \forall i > n_k$,k=1,2, ⋯ , m′

Then $w = \sum_{k=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} T^i {T^*}^j (\sum_{l\in\Lambda} a_{ik} b_{kl} v_l)$

$$= \sum_{j=0}^{m} \sum_{i=0}^{n} T^i {T^*}^j (\sum_{l\in\Lambda} \sum_{k=1}^{m'} a_{ik}\, b_{kl} v_l)$$

$= \sum_{j=0}^{m} \sum_{i=0}^{n} T^i {T^*}^j (\sum_{l\in\Lambda} c_{il} v_l)$, where $c_{il} = \sum_{k=1}^{m'} a_{ik} b_{kl}$

Thus $w = \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} T^i {T^*}^j v_l$ ∎

The definition and proposition that mention it above are explained in the next examples.

Examples 2-16 :

1. Let $\{v_l : l \in \Lambda\}$ be a basis for a normed space V.

(a) Let 0 be the zero operator on V. Recall that $0^0 = I$. If $w \in V_{0,0^*}$ then by proposition (1-12) $= \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} 0^i {0^*}^j v_l$, $c_{il} \in F$. Since $0^* = 0$, $w = \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} 0^i 0^j v_l = \sum_{l\in\Lambda} c_{0l}\, I \,.\, I\, v_l$, $w = \sum_{l\in\Lambda} c_{0l} v_l$

(b) Let $I : V \longrightarrow V$ be the Identity operator on V. If $w \in V_{I,I^*}$ then by proposition (1.2.12) $w = \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} I^i {I^*}^j v_l$. Since $I^* = I$, $\sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} I^i I^j v_l = \sum_{j+i}^{m+n} \sum_{l\in\Lambda} c_{il} I^{i+j} v_l$ put $c_l = \sum_{j+i=0}^{m+n} c_{il}$, then $w = \sum_{l\in\Lambda} c_l\, v_l$

2 . Let $T \in B(H)$, where $B(H)$ space of all bounded linear operator on a Hilbert space H.

(a) Let T be a Self – adjoint operator on H. Then by proposition (1-15)

$$w = \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} T^i T^j v_l$$

(b) Let T be a Normal operator on H. Then by proposition (1-15)

$w = \sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l\in\Lambda} c_{il} {T^*}^i T^j v_l$

Proposition 2-17: Let T and S be two bounded operators on V. Then V_{S,S^*} and V_{T,T^*} are isomorphic R –module if and only if S and T are similar.

Proof:- If V_{S,S^*} is isomorphic to V_{T,T^*} Let $h: V_{S,S^*} \longrightarrow V_{T,T^*}$ be an R –isomorphisim. Thus $h(w_1 + w_2) = h(w_1) + h(w_2)$, for all $w_1, w_2 \in V_{S,S^*}$, $h(P(x,y) \cdot w) = P(x,y) \cdot h(w)$, for all $P \in R$, $w \in V_{S,S^*}$, this mean that h is homomorphisim .Then we can define h as: $h[P(S,S^*)w] = P(T,T^*)h(w)$. If P is a constant polynomial a, $a \in F$, then $h(av) = ah(v)$. Thus h is a linear operator call it again h If $P(x,y) = x + y$. Then $h(P(x,y)w) = P(x,y)h(w), h((x+y)w) = (x+y)\ h(w)$. $h(S+S^*) = (T+T^*)h$, thus $hSh^{-1} + hS^*h^{-1} = h^{-1}Th + h^{-1}T^*h$, Then $hSh^{-1} = T$, $hS^*h^{-1} = T^*$. Then S is similar to T. Conversely, If S and T are similar then there exists an operator h on V such that $h(S+S^*)h^{-1} = T + T^*$ it is easy to cheack that $hP(S,S^*) = P(T,T^*)h$, for all $P \in R$ … … (1) . Define $h': V_{S,S^*} \to V_{T,T^*}$. By $h'[P(S,S^*)v] = P(T,T^*)h(v)$ ……….(2)

If $P_1(S,S^*)v_1 = P_2(S,S^*)v_2$. Then $h[P_1(S,S^*)v_1] = h[P_2(S,S^*)v_2]$ (since h operator). Therefore by (1) $P_1(T,T^*)h(v_1) = P_2(T,T^*)h(v_2)$, then by (2) $h'[P_1(S,S^*)v_1] = h'[P_2(S,S^*)v_2]$. Thus h' is well define. If $h'[P(S,S^*)v] = 0$, then $P(T,T^*)h(v)=0$. Thus by (1) $hp(S,S^*)v = 0$ but h is invertible then $p(S,S^*)\,v = 0$. Therefore h' is 1-1 Let $P(T,T^*)v \in V_{T,T^*}$ .Since $v \in V$, then $h^{-1}(v) \in V$ and $P(S,S^*)h^{-1}(v) \in V_{S,S^*}$. Now, $h'[P(S,S^*)h^{-1}(v)] = P(T,T^*)hh^{-1}(v) = P(T,T^*)v$. Thus h' is onto. Note $h'[P(S,S^*)v] = h[P(S,S^*)v]$. But h is an operator on V, thush is an R −homomorphism. Therefore V_{S,S^*} is isomorphic to V_{T,T^*} .

Remark 2-18: V_{I,I^*} is a finitely generated R −module if and only if V is a finite dimensional normed space .

proof:-Let V be a normed space such that V_{I,I^*} is a finitly generated R −module with generators$\{u_1,u_2,\dots,u_m\}$. We prove by contradiction .Suppose that V is not finite dimensional Let $\{e_\alpha : \alpha \in \Lambda\}$ be a basis for V . Since $u_l \in V$, then $u_l = \sum_{k\in\Lambda} c_k e_k$, l =1,2,…,m Thus V_{I,I^*} can be generated by a finite number of elements of the set $\{e_\alpha : \alpha \in \Lambda\}$, say, $\{e_1,e_2,\dots,e_n\}$. Therefore if $K > n$ then

$e_k=\sum_{t=1}^{n} P_t \,.\, e_t$, where $P_t(x,y)=\sum_{i=0}^{m'} (\sum_{j=0}^{k_t} a_{tj}\ x^j)y^i$, where $P_t(x)=\sum_{j=0}^{k_t} a_{tj}\ x^j$. Hence $P_t \,.\, e_t=\sum_{i=0}^{m'} (\sum_{j=0}^{k_t} a_{tj}\ x^j)y^i \,.\, e_t=\sum_{t=0}^{k_t} a_{tj} e_t$. Put $a_t=\sum_{t=0}^{k_t} a_{tj}$, then $P_t \,.\, e_t=a_t \,.e_t$,t=1,2,…,n . Therefore, $e_k=\sum_{t=1}^{n} a_t e_t$, which is a contradiction. Thus V is a finite dimensional normed space. Assume V is an n−dimensional normed space with basis $\{ v_1, v_2, \dots , v_n \}$. Let $w \in V_{I,I^*}$ by Ex(1-b) $w=\sum_{l=1}^{n} c_l v_l$. This shows that V_{I,I^*} is a finitely generated R −module.

The next remark refers that one side is true, where V is a finite dimensional normed space and T bounded linear operator on V.

Remark 2-19: Let V be a finite dimensional normed space, and T be bounded operator on V, then V_{T,T^*} is a finitely generated $R-$ module.

The following proposition give a sufficient condition for the converse.

Proposition 2-20: If T is of finite rank, and V_{T,T^*} is finitely generated, then V is finite dimensioal.

Proof: Let $K=K(T\,T^*)=\{w \in V: TT^*w=0\}$ it is clear that K is an invariant subspaces of V , and $TT^*\,V \cong \frac{V}{K}$. Suppose V is not finite dimensional, since T is finite rank then TT^*V is finite dimensional, thus K must be infinite dimensional. But K is an invariant subspace of V, then the submodule K_{T,T^*} is generated by the set $\{T^i T^{*^j} w_l : l \in \Lambda; i=0,1,2,\cdots; j=0,1,2,\cdots\}$ where $\{w_l : l \in \Lambda\}$ is a basis for K . But $w_l \in K$,means that $T\ T^* w_l=0$. Hence the restriction of $T\,T^*$ on K is the zero operator, 0 . Thus $K_{T,T^*}=K_{0,0^*}$. Therefore K_{T,T^*} cannot be finitely generated (see (1-16)). But K_{T,T^*} is a submodule of V_{T,T^*} and V_{T,T^*} is finitely generated, this mean infinitely generated contain in finitely generated .This contradiction shows that V is finite dimensional.

The following shows that V_{T,T^*} is Noetherian (Artinian) R −module if V is finite dimensional .

Proposition 2-21: If V is a finite dimensional normed space, and T is a bounded operator on V, then V_{T,T^*} is Noetherian R −module .

Proof :- Let $K_1 \subseteq K_2 \subseteq K_3 \subseteq \ldots$ be an ascending sequence of submodules of V_{T,T^*}. Then it is an ascending sequence of subspaces of V . But V is finite dimensional, thus this sequence is finite .Hence V_{T,T^*} is a Noetherian R −module.

Proposition 2-22 : If V is a finite dimensional normed space , and T is a bounded operator on V , then V_{T,T^*} is Artinian R −module .

Proof :- Let $K_1 \supseteq K_2 \supseteq K_3 \supseteq \ldots$ be a descending sequence of submodules of V_{T,T^*}. Then it is a descending sequence of subspaces of V . But V is finite dimensional, thus this sequence is finite .Therefor V_{T,T^*} is an Artinian R −module. ■

Now we give some results about the R −module of the Unilateral shift operator, starting with the following.

Definition 2-23 : (*- algebraic operator)

An operator $\mathrm{T} \in \mathrm{B(H)}$ is said to be *- algebraic operator if there exists non-zero polynomial of two variables P such that $\mathrm{P(T, T^*)x} = 0$ for all $x \in H. x \in H$ is called ***-** algebraic element if there exists non zero polynomial of two variables P such that $\mathrm{P(T, T^*)x} = 0$.

Proposition 2-24 : Let $\mathrm{T} \in \mathrm{B(H)}$ and $\mathrm{A} = \mathrm{A(T, T^*)}$ be the set of all *- algebraic elements , then As a subspace of H.

Proof:- Clear $A \neq \emptyset$. Let $\mathrm{x, y} \in \mathrm{A}$ then there exists non-zero polynomials p , q in R such that $\mathrm{P(T, T^*)x} = 0$ and $\mathrm{q(T, T^*\ )\ y} = 0$, it is clear that

$P(T,T^*)q(T,T^*)(x+y)=0$. Since $R = F[x,y]$ is an integral domain, then $pq \neq 0$, therefore $x+y \in A$. If $a \in F$ then $P(T,T^*)ax = aP(T,T^*)x = 0$ thus $ax \in A$ .Therefore A is a subspace of H. This subspace call it the subspace of $*-$algebraic elements of the operator T.

Proposition 2-25 : Let T be an operator on H, then $A_{T,T^*} = \tau(H_{T,T^*})$

Proof:- Let $0 \neq w \in A_{T,T^*}$. then $w = \sum_{i=0}^{n} P_i \; x_i$ for some $P_i \in R$, $x_i \in A$ for all i. There exists $q_i \neq 0$ in R such that $q_i(T,T^*)x_i = 0$. Hence $q(T,T^*)\,w = q.w = 0$ where $q = q_1 q_2 \cdots q_n$.Thus $w \in \tau(H_{T,T^*})$. On the other hand , let $y \in \tau(H_{T,T^*})$,then there exists $P \neq 0$ in R

Such that $P.y = 0$, therefore $P(T,T^*)y = 0$. This implies $y \in A$, thus $y \in A_{T,T^*}$. Therefore $A_{T,T^*} = \tau(H_{T,T^*})$

Preposition 2-26: H_{T,T^*} is a faithful $R-$module if and only if T is not $*-$algebraic operator.

Proof:- Let $P \in R$ such that $P(T,T^*)x = 0$ for all $x \in H$.

Then $P.x = 0$ for all $x \in H$. Thus $P.x = 0$ for all $x \in H_{T,T^*}$, hence, $P \in ann(H_{T,T^*})$. Therefor $P = 0$ and T is not $*-$algebraic operator. Conversely, let $P \in ann\,(H_{T,T^*}$. Then $P.x = 0$ for all $x \in H_{T,T^*}$,thus $P(T,T^*)x = 0$ for all $x \in H$. If T is not $*-$algebraic operator, then $P = 0$.Therefor $H_{T,T^*}$is faithful.■

Theorem 2-27 : Let U be the Unilateral shift operator on H. Then H_{U,U^*} is a cyclic R- module .In particular a free R-module.

Proof:- Let $w \in H_{U,U^*}$,then $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^i {U^*}^j e_l$. Since $U^* = B$, $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^i B^j e_l$, $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^i e_{l-j}$ .By remark (1.- 9) (2), $w = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} U^{i+l-1} U^{-j} e_1$. By (1-19) remark 3.

Thus $w = P.e_1$,where $P(x,y) = \sum_{l=1}^{m'} \sum_{j=0}^{m} \sum_{i=0}^{n} a_{il} x^{i+l-1} y^{j}$. Therefore H_{U,U^*} is cyclic R-module generated by e_1.Thus H_{U,U^*} is a free R-module. [7]

Corollary 2-28 : Let U be the Unilateral shift operator on H. Then H_{U,U^*} is a faithful R-module .

Proof: Let $P(x,y) = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} x^i y^j \in ann\,(H_{U,U^*})$. Then $P(x,y).e_1 = 0$ Hence $\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} U^i B^j e_1 = 0$, $\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} U^i (e_{1-j}) = 0$ .By (1-19) remark 2. $\sum_{i=0}^{m} \sum_{j=0}^{n} a_{ij} e_{i-j+1} = 0$. By (1.19) remark 2 . But $e_1, e_2, \dots, e_{m-n+1}$ are linearly Independent . Hence $a_{ij} = 0$, for all $i = 0,1,\dots,m$, $j = 0,1,\dots,n$ Thus P=0 . Therefore H_{U,U^*} is a faithful R-module.

Remark 2-29 [2]: extending to n −variables polynomial as follows :

$R[x_1, x_2, \dots, x_n] = (R[x_1, x_2, \dots, x_{n-1}])[x_n]$. Then we can define a generalized left R - module V . $\Psi: R \times V \longrightarrow V$ define as follows: $\Psi(P, v) = P\,(T_1, T_2, \dots, T_n)v$.

Conclusion : The concepts of bounded linear operator and R-modules have been studied when R is polynomial ring with one variable and two variable and study their properties as follows, If S=$\{v_l : l \in \Lambda\}$ is a basis for V. then each element of V_{T,T^*} can be written in the form $\sum_{j=0}^{m} \sum_{i=0}^{n} \sum_{l \in \Lambda} c_{il} T^i T^{*j} v_l$,where $c_{il} \in F$. The symbol $\sum_{l \in \Lambda}$ means that the sum is taken over a finite subset of Λ and Let U be the Unilateral shift operator on H. Then H_{U,U^*} is a cyclic R- module .In particular a free R-module.

بحث مراجعة حول المؤثرات المقيدة والمقاسات

شيرين عوده دخيل [١] ،سميرة ناجي كاظم [٢]، عهود سعدي الحسنى [٣] و منى جاسم محمد علي [٤]

[١،٢،٤] جامعة بغداد /كلية العلوم للبنات/ قسم الرياضيات

[٣] جامعة بغداد /كلية العلوم/ قسم الحاسبات

المستخلص: الهدف من عملنا هو مراجعة المقاسات والمؤثرات الخطية المحددة. افترض أن V عبارة عن فضاء متجهات معرف على الحقل F. ضع T هو مؤثر خطي. ضع R=F[x] هي حلقة كثيرات الحدود في x مع المعاملات في F. حدد Ø:R×V⟶V معرفة بالشكل $\emptyset(P,v) = P(T)v = P.v$ هذا Ø يجعل V مقاس ايسر على R يرمز له V_T. تم تقديم تعميم لهذا المفهوم، فوضع V هوفضاء متجهات على الحقل F، ووضع T هو B. L. O. ، وافترض أن R=F[x,y] هي حلقة كثيرات الحدود في x,y مع معاملات في F لتكن Ψ:R×V→V معرفة بالشكل $\Psi(P,v) = P.v = P(T,T^*)v$ و V_{T,T^*} هي مقاس. وتمت دراسة بعض خصائص هذه المفاهيم.

الكلمات المفتاحية: المؤثرات الخطية المحددة، الحلقات متعددة الحدود، الفضاء المعياري والمقاسات.

Refrences

[1] Sterling K. Berberian,(1961), "Introduction to Hilbert Space", chelsea. publishing company ,New York ,N.Y.

[2] Heba Hadi Chyad and Akeel Ramadan Mehdi, SAS-Injective Modules, Journal of Al-Qadisiyah for Computer Science and Mathematics, Vol.15, No.2, 2023.

[3] Salma M .Faris , (1994), "Linear Operators and Modules", a master thesis, department Mathematics, College of Science, University of Baghdad, Baghdad., Iraq

[4] Kaplansky I, (1962), "Infinite abelian groups", The University of Michigan press , Ann Arbor

[5] Pawe l Pietrzycki and Jan Stochel, Subnormal nth roots of quasinormal operators are quasinormal, arXiv:2008.04758v3,[math FA], 24 sep 2020

[6] Manal Ali Sagban (2016) "Modules and Bounded Linear operators" M.Sc.Thesis ,College of Science for women, University of Baghdad, Baghdad, Iraq.

[7] Kreyszig Erwin, (1978), "Introductory functional analysis with applications" , John weily and Sons, University of Windsor.

[8] David M. Burton,(1972), Abstract and linear Algebra, University of New Hampshire. Addison-Wesley Publishing Company.

[9] Krutan Rasimi, A Note On A Result About Skew Binormal Operators, Journal of Natural Sciences and Mathematics of UT, Vol. 5, No. 9-10, 2020

[10] Cherifa Chellali , Abdelkader Benali and Israal Shbeil, Some Properties Of (n, m)-Power Hyponormail In Semihilbertian Space, Journal of Science and Arts, Vol. 22, Issue 1, pp. 115-120, 2022.

[11] Rajesh Kumar Pa and Deependra Nigam,(2022) Modern Research and Applications in Mathematical Sciences, Rubicon Publications, England.